A CHARACTERIZATION OF $\aleph_1$-FREE ABELIAN GROUPS AND ITS APPLICATION TO THE CHASE RADICAL

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ABSTRACT

A group A is an $\aleph_1$-free abelian group iff A is a subgroup of the Boolean power $\mathbf{Z}^{(B)}$ for some complete Boolean algebra B. The Chase radical $\nu A = \Sigma\{C \leqq A : \mathrm{Hom}(C, \mathbf{Z}) = 0 \ \&\ C \text{ is countable}\}$. The torsion class $\{A : \nu A = A\}$ is not closed under uncountable direct products.

An abelian group A is $\aleph_1$-free if any countable subgroup of A is free. $\aleph_1$-free abelian groups have been studied by several authors, because Pontrjagin's theorems [22, Section 38] say "A compact abelian group G is connected and locally connected iff the dual group of G is $\aleph_1$-free." In the present paper we give a characterization of $\aleph_1$-free groups which clarifies why $\aleph_1$-free groups are somewhat like torsionless groups, though they are not always torsionless. Using it we investigate the Chase radical. We use the Boolean valued models $V^{(B)}$ from Set Theory [2, 17, 21] and hence we assume the knowledge about them and use the notion and notation in [8] in the context. However, we'll outline direct proofs especially in Remark 4 at the end of this paper. Therefore, the reader can avoid the proofs using Boolean valued models $V^{(B)}$. All groups in this paper are abelian groups and any undefined notion for groups is standard [15]. To state our main results, we define Boolean powers.

For a complete Boolean algebra (cBa) B and a group A, the Boolean power $A^{(B)}$ is the group consisting of functions $f: A \to B$ such that $(f(x) : x \in A)$ is a partition of 1 of B, i.e. $f(x) \wedge f(y) = 0$ for distinct x, y and $\bigvee_{x \in A} f(x) = 1$. For $f, g \in A^{(B)}$, $(f+g)(x) = b$ iff $b = \bigvee_{x=u+v} f(u) \wedge f(v)$. This kind of group has been studied in [1, 6, 7, 8, 10, 11, 18]. If A is countable, we need only countable

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completeness of B to define a Boolean power. In such a case, let $\bar{B}$ be the canonical completion of B; then $A^{(B)}$ is a subgroup of $A^{(\bar{B})}$. Well-known groups $\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$ for cardinals κ of uncountable cofinality are isomorphic to $\mathbf{Z}^{(P\kappa/P_\kappa\kappa)}$, where

$$\mathbf{Z}^{<\kappa} = \{x \in \mathbf{Z}^\kappa : \{\alpha : x(\alpha) \neq 0\} \text{ is of cardinality less than } \kappa\}$$

and $P\kappa$ is the power set of κ and

$$P_\kappa\kappa = \{x \in P\kappa \text{: The cardinality of } x \text{ is less than } \kappa\}.$$

See [5, 10, 11, 24] for those groups.

In Theorem 1 the equivalence of (1) and (2) is due to Kueker [20, Corollary 3.6] and Ellentuck [13].

THEOREM 1. *The following propositions are equivalent for an abelian group A:*

(1) *A is $\aleph_1$-free;*

(2) (Kueker-Ellentuck) *$A^\vee$ is free in $V^{(B)}$ for some complete Boolean algebra* (cBa) *B;*

(3) *$A^\vee$ is torsionless in $V^{(B)}$ for some* cBa *B;*

(4) *A is a subgroup of the Boolean power $\mathbf{Z}^{(B)}$ for some* cBa *B.*

Let νA be the Chase radical, i.e.

$$\nu A = \bigcap\{\mathrm{Ker}(h) : h \in \mathrm{Hom}(A, X) \text{ and } X \text{ is } \aleph_1\text{-free}\}.$$

THEOREM 2. *The following propositions are equivalent for an abelian group A and an element a of A.*

(1) *a belongs to νA;*

(2) *For every* cBa *B and $h \in \mathrm{Hom}(A, \mathbf{Z}^{(B)})$, $h(a) = 0$;*

(3) *For every* cBa *B, $[\![\forall h \in \mathrm{Hom}(A^\vee, \mathbf{Z})(h(a^\vee) = 0)]\!]^{(B)} = 1$;*

(4) *a belongs to $\Sigma\{C \leqq A : \mathrm{Hom}(C, \mathbf{Z}) = 0$ & C is countable$\}$.*

COROLLARY 3. *The following propositions are equivalent for an abelian group A:*

(1) *$\nu A = A$;*

(2) $\mathrm{Hom}(A, \mathbf{Z}^{(B)}) = 0$ *for every* cBa *B;*

(3) $\mathrm{Hom}(A^\vee, \mathbf{Z}) = 0$ *in $V^{(B)}$ for every* cBa *B;*

(4) *$A = \Sigma\{C \leqq A : \mathrm{Hom}(C, \mathbf{Z}) = 0$ and C is countable$\}$.*

COROLLARY 4. *The Chase radical ν satisfies the cardinal condition.*

This answers a question in [14]. See [5, 14] for the definition of the cardinal condition and related notions. In [5, Theorem 5.1] it has been shown that the torsion class

$$\{A : \nu A = A\} = \{Q, \oplus, E\}\{H\},$$
$$\text{where } H = \bigoplus [C : \mathrm{Hom}(C, \mathbf{Z}) = 0 \text{ and } C \text{ is countable}].$$

Corollary 3 implies that $\{A : \nu A = A\} = \{Q, \oplus\}\{H\}$ with the same H.

THEOREM 5. *Let A be a countable group and* $\mathrm{Hom}(A, \mathbf{Z}) = 0$. *Then, there exists a torsionfree group G of rank 1 such that* $\mathrm{Hom}(A, G) = 0$ *and* $\mathrm{Hom}(G, \mathbf{Z}) = 0$.

COROLLARY 6. *The torsion class $\{A : \nu A = A\}$ is not closed under uncountable direct products.*

This answers a question of [5, Section 5] affirmatively.

1. Proofs of Theorems 2 and 3 and related consequences

LEMMA 7 (Pontrjagin [15, Theorem 19.1]). *For a torsionfree group A, A is $\aleph_1$-free iff any subgroup of A of finite rank is free. Consequently, a countable torsionfree group A is free, if any subgroup of A of finite rank is free.*

LEMMA 8. *Let A be a torsionfree group and X a subset of A. Then. $A^\vee$ is torsionfree and the pure closure $\langle X^\vee \rangle_*$ is equal to $(\langle X \rangle_*)^\vee$ in $V^{(B)}$ for a* cBa B.

PROOF. We check the absoluteness of notions concerning Boolean valued models. A is torsionfree iff $\forall a \in A\ \forall n \in \mathbf{Z}$ ($na = 0$ implies $n = 0$ or $a = 0$). For an $a \in A$, a belongs to $\langle X \rangle_*$ iff there exist $n_0 \cdots n_k \in \mathbf{Z}$ and $a_1 \cdots a_k \in X$ such that $n_0 \neq 0$ and $n_0 a + n_1 a_1 + \cdots + n_k a_k = 0$. Since the notion of finite subsets is absolute, the lemma holds.

PROOF OF THEOREM 1. (1)$\to$(2). Let B be a cBa for which A is countable in $V^{(B)}$. For example, let B be the Boolean algebra consisting of all regular open subsets of A^N, where A is discrete and A^N is endowed with the product topology [17, 21]. Then, $A^\vee$ is $\aleph_1$-free and hence a free group of countable rank in $V^{(B)}$ by Lemmas 7 and 8.

(2)$\to$(3). Clear.

(3)$\to$(4). If A is torsionless in $V^{(C)}$ for a cBa C, taking a large enough I we get an $i \in V^{(C)}$ such that

$$[\![i : A^{\vee} \to \mathbf{Z}^{I^{\vee}} \text{ is an injective homomorphism}]\!]^{(C)} = 1.$$

Then, the mapping a to $i(a^{\vee})$ is an injective homomorphism from A into $(\mathbf{Z}^{I^{\vee}})^{\wedge}$. Since $\mathbf{Z} = \mathbf{Z}^{\vee}$ in $V^{(C)}$, $\mathbf{Z}^{\wedge} \simeq \mathbf{Z}^{(C)}$ and hence $(\mathbf{Z}^{I^{\vee}})^{\wedge} \simeq (\mathbf{Z}^{(C)})^{I}$. Let B be the direct product C^{I} of copies of the Boolean algebra C. Then, $\mathbf{Z}^{(B)} \simeq (\mathbf{Z}^{(C)})^{I}$. Hence, A is isomorphic to a subgroup of $\mathbf{Z}^{(B)}$.

(4)→(1). It is enough to show that $\mathbf{Z}^{(B)}$ is $\aleph_1$-free for any cBa B. Let $f_1, \ldots, f_n$ be elements of $\mathbf{Z}^{(B)}$. There exists a partition $(b_m : m \in N)$ of 1 of B such that $b_m \wedge f_i(a) \neq 0$ iff $b_m \leqq f_i(a)$ for each $m \in N$, $a \in \mathbf{Z}$, $1 \leqq i \leqq n$. Let

$$S = \{ f \in \mathbf{Z}^{(B)} : b_m \wedge f(a) \neq 0 \text{ iff } b_m \leqq f(a) \text{ for any } m \in N,\ a \in \mathbf{Z} \}.$$

Then, S is isomorphic to $\mathbf{Z}^{N}$ or free group of finite rank and $\langle f_1 \cdots f_n \rangle_*$ is a subgroup of S. Since $\mathbf{Z}^{N}$ is $\aleph_1$-free [15, Theorem 19.2] $\langle f_1 \cdots f_n \rangle_*$ is free and hence $\mathbf{Z}^{(B)}$ is $\aleph_1$-free by Lemma 7.

For embedding an $\aleph_1$-free group A to $\mathbf{Z}^{(B)}$, let C be $\mathrm{RO}(A^{N})$, i.e. the cBa indicated in the proof of (1)→(2), then $B(=C^{N})$ is isomorphic to $\mathrm{RO}(A^{N})$. Hence, one might think that B can be taken as not such a complicated one. However, since any cBa can be completely embedded into such a kind of cBa's [19], we cannot say that $\mathrm{RO}(A^{N})$ has a simple structure to embed A. If B is a cBa and completely distributive, then B is atomic [23, 25.2] and so $\mathbf{Z}^{(B)}$ is a direct product of $\mathbf{Z}$. Hence, if we can embed A into $\mathbf{Z}^{(B)}$ where B has high distributivity, then it means that A is near to be torsionless. More precisely, let κ be the cardinal of A and B a κ-representable Boolean algebra [23, Section 29]. If A is a subgroup of $\mathbf{Z}^{(B)}$, then A is torsionless.

Proof of Theorem 2. (1)→(2). Clear by the equivalence of (1) and (4) of Theorem 1.

(2)→(3). Suppose the negation of (3), then by the maximum principle [2] there exists an $h \in (\mathrm{Hom}(A^{\vee}, \mathbf{Z}))^{\wedge}$ such that $[\![h(a^{\vee}) \neq 0]\!]^{(B)} \neq 0$. The mapping a to $h(a^{\vee})$ is a homomorphism from A to $\mathbf{Z}^{(B)}$ under the isomorphism $\mathbf{Z}^{\wedge} \simeq \mathbf{Z}^{(B)}$ and $h(a^{\vee}) \neq 0$.

(3)→(4). Suppose the negation of (4), then there exists an $a_0 \in A$ such that for any countable subgroup C of A containing a_0,

$$a_0 \notin \bigcap \{ \mathrm{Ker}(h) : h \in \mathrm{Hom}(C, \mathbf{Z}) \}$$

by the proof of Stein's lemma [15, Corollary 19.3]. Let P be the set

$\{\sigma$: dom σ is a countable subgroup of A & $a_0 \in$ dom σ &
$\sigma \in \mathrm{Hom}(\mathrm{dom}\ \sigma, Z)$ & $\sigma(a_0) \neq 0$ & σ can be extended
to any countable subgroup which includes dom $\sigma\}$.

The partial ordering of P is defined by the extension as functions, i.e. $\sigma \leqq \tau$ iff σ is an extension of τ. Since the proof of the nonemptiness of P is similar to the following one, we omit it. Let

$$D_x = \{\sigma \in P : x \in \mathrm{dom}\ \sigma\} \quad \text{for each } x \in A.$$

Now, we show that D_x is dense in P. Suppose the negation, then there exists a $\tau \in P$ with the following: For any $h \in \mathrm{Hom}(\mathrm{dom}\ \tau + \langle x \rangle, \mathbf{Z})$ which extends τ, there exists a countable subgroup C_h such that h cannot be extended onto C_h. Since such h is determined by the value at x, there exists only countably many such h's. Hence, there exists a countable subgroup C such that any such h cannot be extended to C. However, τ can be extended to C and the restriction of the extension to dom $\tau + \langle x \rangle$ must be one of the above h, which is a contradiction. Now, let B be the cBa related to P, i.e. the Boolean algebra of all the regular open subsets of P where $U_\sigma = \{\tau \in P : \tau \leqq \sigma\}$ is a basic open set for each $\sigma \in P$. Then, for the generic filter G of $P^\vee$, $\bigcup G$ is a homomorphism from $A^\vee$ to $\mathbf{Z}$ such that $\bigcup G(a^\vee) \neq 0$ in $V^{(B)}$.

(4)→(1). Clear.

Since the direct proof of (2)→(4) is not so complicated in comparison with that of (1)→(4) of Theorem 1, we present it here. Suppose the negation of (4). We define a map $\varphi : A \to \mathbf{Z}^{(B)}$, where P, B and U_σ are as in the proof of (3)→(4). For $x \in A$, $n \in \mathbf{Z}$ let

$$\varphi(x)(n) = \bigvee\{U_\sigma : \sigma \in P \ \&\ x \in \mathrm{dom}\ \sigma \ \&\ \sigma(x) = n\}.$$

If $\sigma(x) \neq \tau(x)$ holds for $\sigma, \tau \in P$, then $U_\sigma \wedge U_\tau = 0$. $\bigvee\{U_\sigma : \sigma \in D_x\} = 1$ for each $x \in A$, as shown in the proof. Therefore, $(\varphi(x)(n) : n \in \mathbf{Z})$ is a partition of 1 for each $x \in A$. $\varphi(a_0) \neq 0$ is clear. The following fact implies that φ is a homomorphism:

$$\forall \rho \in U_\sigma \cap U_\tau (\sigma, \tau \in D_x \cap D_y \text{ imply } \rho(x+y) = \rho(x) + \rho(y)).$$

Now, Corollary 3 follows from Theorem 2. In [3] it was shown that $\nu\nu = \nu$ holds and for a countable group C, $\nu C = C$ iff $\mathrm{Hom}(C, \mathbf{Z}) = 0$. Hence, Corollary 4 follows from Theorem 2.

LEMMA 9. *Let κ be an uncountable regular cardinal and G a group of cardinality less than κ. For a group A the following* (1) *and* (2) *are equivalent*:

(1) $A = \Sigma\{X \leqq A : \mathrm{Hom}(X, G) = 0$ *and the cardinality of X is less than* $\kappa\}$;

(2) *For any $a \in A$ and nonzero $g \in G$, there exists a subgroup C of cardinality less than κ such that $a \in C$ and for any $h \in \mathrm{Hom}(C, G)$, $h(a) \neq g$.*

PROOF. The implication (1)→(2) is clear. Let C_{ag} be a subgroup which satisfies the condition of (2) for each $a \in A$ and nonzero $g \in G$. Let

$$C_a = \Sigma\{C_{ag} : g \in G \,\&\, g \neq 0\},$$

then the cardinality of C_a is less than κ. Define $E_1 = \langle a \rangle$ and $E_{n+1} = \Sigma\{C_x : x \in E_n\}$ and let $C = \Sigma_{n \in N} E_n$. Then, $\mathrm{Hom}(C, G) = 0$ and the cardinality of C is less than κ and C contains a.

By Lemma 9 we can relax the condition of supercompactness to that of compactness in [4, Theorem 2.3]. This answers a question of [4] affirmatively.

Next we generalize a part of Theorem 2 and Corollary 3.

THEOREM 10. *Let G be a countable group, $R_G^* A = \bigcap\{\mathrm{Ker}(h)$: $h \in \mathrm{Hom}(A, G)\}$ and $R_G^* A = \bigcap\{\mathrm{Ker}(h) : h \in \mathrm{Hom}(A, G^{(B)})$ & B is a cBa$\}$. Then, $R_G^* A = \Sigma\{R_G C : C$ is a countable subgroup of $A\}$. Consequently, $\mathrm{Hom}(A, G^{(B)}) = 0$ for every cBa B iff*

$$A = \Sigma\{C \leqq A : C \text{ is countable } \& \ \mathrm{Hom}(C, G) = 0\}.$$

PROOF. Let C be a countable group, $h \in \mathrm{Hom}(C, G^{(B)})$ for a cBa B and $h(c) \neq 0$ for a $c \in C$. Since C is countable, by the so-called Rasiowa–Sikorski Lemma [2, p. 5; 21], there exists an ultrafilter F of B such that $h(c)(g^*) \in F$ for some nonzero $g^* \in G$ and for each $x \in C$ there exists a unique $g \in G$ so that $h(x)(g) \in F$. Define $\varphi(x) = g$ by $h(x)(g) \in F$. Then, φ is a nonzero homomorphism from C to G, since F is a filter and $h(c)(g) \in F$. Hence, $R_G C = R_G^* C$ and so

$$\Sigma\{R_G C : C \text{ is a countable subgroup of } A\} \subset R_G^* A.$$

For the converse inclusion, the same proof of (2)→(4) of Theorem 2 goes.

For the second proposition, it is enough to observe that

$$A = \Sigma\{R_G C : C \text{ is a countable subgroup of } A\}$$

iff $A = \Sigma\{C \leqq A : \mathrm{Hom}(C, G) = 0 \,\&\, C$ is countable$\}$ by Lemma 9.

2. Proofs of Theorem 5 and Corollary 6 and remarks

Any torsionfree group of rank 1 can be identified by its type t and isomorphic to a subgroup of the rational group $\mathbf{Q}$ [16, Section 85]. Let R_t be a torsionfree group of rank 1 with its type t.

PROOF OF THEOREM 5. Starting from the type t_0 containing the characteristic $(1, 1, 1, \ldots)$, we get a sequence of types $\{t_\alpha : \alpha < \kappa\}$ with the following: κ is a regular and uncountable cardinal; $0 < t_\beta < t_\alpha$ for $\alpha < \beta < \kappa$; $0 = \inf\{t_\alpha : \alpha < \kappa\}$. This can be done, because the order structure between 0 and t_0 is isomorphic to the quotient Boolean algebra of $P(N)$ modulo finite subsets and so we cannot reach 0 by a countable sequence [17, p, 261]. Let

$$r_\alpha A = \bigcap\{\mathrm{Ker}(h) : h \in \mathrm{Hom}(A, R_{t_\alpha})\} \quad \text{for each } \alpha < \kappa.$$

Then, $r_\alpha A \leqq r_\beta A$ for $\alpha \leqq \beta$. Since A is countable, there exists a γ such that $r_\alpha A = r_\gamma A$ for every $\alpha \geqq \gamma$. Let $C = A / r_\gamma A$, then $r_\alpha C = 0$ for every $\alpha \geqq \gamma$, i.e. C is isomorphic to a subgroup of $(R_{t_\alpha})^N$. Now, we shall show that C is free. Let F be a subgroup of C of finite rank. Since $\mathrm{Hom}(F, \mathbf{Q})$ is countable, the set of types $\{t : R_t \text{ is a homomorphic image of } F\}$ is countable. Hence, there exists a $\beta \geqq \gamma$ such that $h(F) = 0$ or $h(F) \simeq \mathbf{Z}$ for any $h \in \mathrm{Hom}(F, R_{t_\beta})$. Since F is isomorphic to a subgroup of $(R_{t_\beta})^N$, F is isomorphic to a subgroup of $\mathbf{Z}^N$ and hence is free by [15, Theorem 19.2]. Therefore, C is free by Lemma 7. By the assumption, $\mathrm{Hom}(C, \mathbf{Z}) = 0$ and so $C = 0$. i.e. $A = r_\gamma A$ and $\mathrm{Hom}(A, R_{t_\gamma}) = 0$. The group R_{t_γ} is a desired one.

As remarked in [5, Section 5], the torsion class $\{X : \nu X = X\}$ is closed under countable direct products. However, it is not closed under uncountable direct products as Corollary 6 shows and also conjectured in [5, p. 101].

PROOF OF COROLLARY 6. Let $A = \Pi_{\alpha<\kappa} R_{t_\alpha}$, where R_{t_α} $(\alpha < \kappa)$ are the ones defined in the proof of Theorem 5 and $a^* \in A$ such that $a^*(\alpha) \neq 0$ for any $\alpha < \kappa$. Clearly, $\nu R_{t_\alpha} = R_{t_\alpha}$ for each $\alpha < \kappa$. If $\mathrm{Hom}(C, \mathbf{Z}) = 0$ for a countable subgroup of A, then C cannot contain a^* as shown in the proof of Theorem 5. Hence, a^* does not belong to νA by Theorem 2, i.e. $\nu A \neq A$.

REMARKS. (1) A group $\mathbf{Z}^{(B)}$ has a canonical maximal free pure subgroup $\bar{\mathbf{Z}}^{(B)}$ $(= \{x : x(n) = 0$ for all but finite $n \in \mathbf{Z}\})$. Since $\bar{\mathbf{Z}}^{(B)}$ is isomorphic to the group consisting of all integer valued continuous functions from the stone space of B, the rank of $\bar{\mathbf{Z}}^{(B)}$ is equal to the topological weight of the stone space of B [12, Corollary 2.5].

(2) As in the remark following the proof of Theorem 1, any $\aleph_1$-free group can be embedded into a group $\mathbf{Z}^{(B)}$ where $B = \mathrm{RO}(I^N)$ for some I. Since such a B has a system $\{b_{n0}, b_{n1} : n \in N\}$ such that $b_{n0} \vee b_{n1} = 1$ and $\bigwedge_{n\in N} b_{nf(n)} = 0$ for any $f: N \to \{0, 1\}$, B has no countably complete maximal filter. Hence, $\mathrm{Hom}(\mathbf{Z}^{(B)}, \mathbf{Z}) = 0$ by [8, Theorem 1].

(3) The size of κ in the proof of Theorem 5 has been studied in set theory [17, 21] and in some models of set theory κ is ω_1 even though $2^{\aleph_0}$ is a larger regular cardinal.

(4) After the completion of this paper, the author has found other proofs of the implication (1)$\to$(4) of Theorem 1 and the equivalence of (1) and (4) of Theorem 2, using reduced products [4]. Here we outline them. Dugas [4] proved that a group A is $\aleph_1$-free iff A is isomorphic to a subgroup of a reduced product of the group $\mathbf{Z}$ by a countably complete filter. Since a reduced power of a countable group by a countably complete filter is isomorphic to a Boolean power by a countably complete Boolean algebra [11], we can show the implication (1)$\to$(4) of Theorem 1. For the equivalence of (1) and (4) of Theorem 2, what we need are a reduced product X of $\mathbf{Z}$ and a homomorphism $h: A \to X$ for an $a_0 \notin \Sigma\{C \leqq A : \mathrm{Hom}(C, \mathbf{Z}) = 0 \ \& \ C$ is countable$\}$ ($= \Sigma\{R_{\mathbf{Z}}C : C$ is a countable subgroup of $A\}$) so that $h(a_0) \neq 0$. Let I be the set of all countable subsets of A. For each $Y \in I$, let $h_Y : \langle Y \rangle \to \mathbf{Z}$ be a homomorphism so that $h_Y(a_0) \neq 0$ if $a_0 \in \langle Y \rangle$ and F the canonical countably complete filter of I generated by $\{Y \in I : a \in Y\}$ $(a \in A)$. Then, there exists canonical homomorphisms $i: A \to \Pi_{Y\in I}\langle Y \rangle / F$ and $h^* : \Pi_{Y\in I}\langle Y \rangle/F \to \Pi_{Y \in I} \mathbf{Z}/F$, where the reduced product $\Pi_{Y\in I}\langle Y\rangle/F$ is the quotient group of $\Pi_{Y\in I}\langle Y\rangle$ by the subgroup $\{f \in \Pi_{Y\in I}\langle Y\rangle : \{Y : f(Y) = 0\} \in F\}$ and h^* is induced by $(h_Y : Y \in I)$. Then, $h^* \cdot i$ and $\Pi_{Y\in I}\mathbf{Z}/F$ are the desired h and X respectively. In addition we remark the following fact can be proved using reduced products [9].

For a radical R and a regular cardinal κ, let

$$R^{[\kappa]}A = \Sigma\{RC : C \text{ is a subgroup of } A \text{ of cardinality less than } \kappa\}$$

[5]. Then, $R^{[\kappa]}$ is also a radical, i.e. $R^{[\kappa]}(A/R^{[\kappa]}A) = 0$ for every A.

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